

On g -Extra Connectivity of Hypercube-like Networks

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Abstract

Given a connected graph G and a non-negative integer g , the g -extra connectivity $\kappa_g(G)$ of G is the minimum cardinality of a set of vertices in G , if it exists, whose deletion disconnects G and leaves each remaining component with more than g vertices. This paper focuses on the g -extra connectivity of hypercube-like networks (HL-networks for short) which includes numerous well-known topologies, such as hypercubes, twisted cubes, crossed cubes and Möbius cubes. All the known results suggest the equality $\kappa_g(X_n) = f_n(g)$ holds, where X_n is an n -dimensional HL-network, $f_n(g) = n(g+1) - \frac{g(g+3)}{2}$, $n \geq 5$ and $0 \leq g \leq n-3$? Some authors also attempted to prove this equality in general. In this paper, we construct a subfamily of an n -dimensional HL-network with g -extra connectivity greater than $f_n(g)$ which implies that the above equality does not hold in general. We also prove that for $n \geq 5$ and $0 \leq g \leq n-3$, $\kappa_g(X_n) \geq f_n(g)$ always holds. This enables us to give a sufficient condition for the equality $\kappa_g(X_n) = f_n(g)$, which is then used to determine the g -extra connectivity of HL-networks for some small g or the g -extra connectivity of some particular subfamily of HL-networks. As a result, a short proof for the main results in [Journal of Computer and System Sciences 79 (2013) 669–688].

Keywords HL-network, extra connectivity, reliability, Cayley graph.

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1 Introduction

With the rapid development of VLSI technology and software technology, a multiprocessor system may contain hundreds or even thousands of nodes. With the continuous increase in the size of multiprocessor systems, the complexity of a system can adversely affect its fault tolerance and reliability. To the design and maintenance purpose of multiprocessor systems, appropriate measures of reliability should be found.

In a multiprocessor system, processors are connected based on a specific interconnection network. An interconnection network is usually represented by a graph in which vertices represent processors and edges represent links between processors. The traditional connectivity is an important factor for measuring the reliability of an interconnection network, which can correctly reflects the fault tolerance of systems with few processor. However,

it always underestimates the resilience of large networks. The discrepancy incurred is because events whose occurrence would disrupt a large network after a few processor failures are highly unlikely, therefore, the disruption envisaged occurs in a worst case scenario. Motivated by the shortcomings of the traditional connectivity, Harary [16] introduced the concept of conditional connectivity.

Let G be a connected undirected graph, and \mathcal{P} a graph-theoretic property. Harary [16] defined the conditional connectivity $\kappa(G; \mathcal{P})$ as the minimum cardinality of a set of vertices, if any, whose deletion disconnects G and every remaining component has property \mathcal{P} . Subsequently, Fábrega and Fiol [13] investigated the following kind of conditional connectivity. A subset of vertices S is said to be a *cutset* if $G - S$ is not connected. A cutset S is called an R_g -cutset, where g is a non-negative integer, if every component of $G - S$ has at least $g + 1$ vertices. If G has at least one R_g -cutset, the *g -extra connectivity* of G , denoted by $\kappa_g(G)$, is then defined as the minimum cardinality over all R_g -cutsets of G . In other words, $\kappa_g(G) = \kappa(G; \mathcal{P}_g)$, where \mathcal{P}_g denotes that every remaining component has more than g vertices.

Obviously, $\kappa_0(G) = \kappa(G)$ for any connected graph G that is not a complete graph. Therefore, the g -extra connectivity can be regarded as a general form of the classical connectivity that provides measures that are more accurate for reliability and fault tolerance for large-scale parallel processing systems. Regarding the computational complexity of the problem, based on thorough research, no polynomial-time algorithm has been presented to compute κ_g for a general graph [6]; nor has there been any tight upper bound for κ_g [10]. The problem of determining the g -extra connectivity of numerous networks has received a great deal of attention in recent years. For more results regarding g -extra connectivity, see, for example, [2, 4, 6, 10, 11, 12, 13, 19, 23, 28, 29, 30]. It is worthwhile to mention that different types of generalized connectivity such as g -extra connectivity have many applications. One of them is the conditional diagnosability, which was firstly proposed by Lai et al. [18]. For extensive study, the readers may also refer to [5, 14, 26].

The *hypercube-like networks* (*HL-networks* for short) are defined recursively as follows:

$$\mathbb{L}_0 = \{K_1\} \text{ and } \mathbb{L}_n = \{G_0 \oplus G_1 \mid G_0, G_1 \in \mathbb{L}_{n-1}\},$$

where the symbol “ \oplus ” represents the *perfect matching operation* that connects G_0 and G_1 using some perfect matching, denoted by $PM(G)$. It is obvious that $\mathbb{L}_1 = \{K_2\}$, $\mathbb{L}_2 = \{C_4\}$, and $\mathbb{L}_3 = \{Q_3, G(8, 4)\}$, where C_4 is a cycle of length 4, and Q_3 and $G(8, 4)$ are depicted as Figure (1). (Some authors also use the term BC-networks instead [14]. In

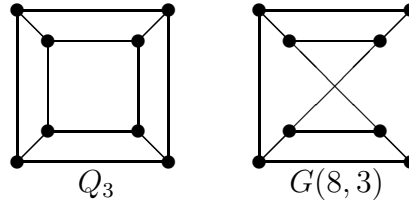


Figure 1: Two 3-dimensional HL-networks

this paper, we follow [22] to use the term HL-networks.) Numerous well-known topologies,

such as hypercubes [21], crossed cubes [9], Möbius cubes [8], twisted cubes [17], varietal cubes [7] etc. belong to the class of HL-networks.

For a positive integer n , let $f_n(g) = n(g+1) - \frac{1}{2}g(g+3)$ be a function of g . From [28], we know that $\kappa_g(Q_n) = f_n(g)$ for $n \geq 4$ and $0 \leq g \leq n-3$. Let G be an n -dimensional HL-network. Fan et al. proved that $\kappa_0(G) = f_n(0) = n$ (see [14]), and $\kappa_1(G) = f_n(1) = 2n-2$ for $n \geq 3$ (see [29]). Xu et al. [25] proved that $\kappa_2(G) = f_n(2) = 3n-5$ for $n \geq 8$ and $\kappa_2(G) \geq 3n-5$ for $5 \leq n \leq 7$. Recently, Chang and Hsieh [4] improved Xu's result by showing that $\kappa_2(G) = f_n(2) = 3n-5$ for $n \geq 5$, and they also obtained that $\kappa_3(G) = f_n(3) = 4n-9$ for $n \geq 6$. The facts listed above provide a strong motivation for studying the following problem.

Problem A For any $G \in \mathbb{L}_n$, does $\kappa_g(G) = f_n(g)$ hold for $0 \leq g \leq n-3$?

In this paper, by analyzing the structure of an n -dimensional HL-network G with at most $f_n(g)$ faulty vertices, where $0 \leq g \leq n-3$ and $n \geq 5$, we give a lower bound on $\kappa_g(G)$, namely, $\kappa_g(G) \geq f_n(g)$. Furthermore, we give a sufficient condition for the equality $\kappa_g(G) = f_n(G)$. Using this condition, we first present a short proof for Chang and Hsieh's results [4]. Then we investigate the g -extra connectivity of a particular subfamily of HL-networks, namely, varietal hypercubes VQ_n . Let $n = 3s + t$ with $s \geq 3$ and $0 \leq t \leq 2$. This study shows that $\kappa_g(VQ_n) = f_n(g)$ for $0 \leq g \leq n-s$. At last, we construct a subfamily of HL-networks to show that the inequality $\kappa_g(G) > f_n(g)$ can hold, and so a negative answer to Problem A is given.

2 Preliminaries

Throughout this paper only undirected simple connected graphs without loops and multiple edges are considered. Unless stated otherwise, we follow Bondy and Murty [3] for terminology and definitions.

2.1 Fundamental graph and group terminologies

A *graph* $G = (V, E)$ is comprised of a vertex set V and an edge set E , where V is a finite set and E is a subset of $\{\{u, v\} \mid u, v \in V, u \neq v\}$. Two vertices, u and v , are *adjacent* if $\{u, v\} \in E$, and u and v are the end-vertices of $\{u, v\}$. A *subgraph* of G is a graph $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. We use $H \subseteq G$ to denote that H is a subgraph of G . Given a vertex set $U \subseteq V$, the subgraph of G induced by U is the graph $G[U] = (U, E')$, where $E' = \{\{u, v\} \in E \mid u, v \in U\}$. For a set of vertices and/or edges, denoted by F , in G , the notation $G - F$ represents a subgraph of G obtained by deleting all the elements in F from G . The components of a graph G are its maximal connected subgraphs.

A k -*path* $P_k = v_0 v_1 \dots v_k$ for $k \geq 1$ in a graph G is a sequence of distinct vertices such that any two consecutive vertices are adjacent; we call v_0 and v_k the end-vertices of the path. A k -*cycle* $C_k = (v_1, v_2, \dots, v_k, v_1)$ for $k \geq 3$ is a sequence of vertices in which any two consecutive vertices are adjacent, where v_1, v_2, \dots, v_k are all distinct. A complete

graph K_n is a graph comprised of n pairwise adjacent vertices. A complete bipartite graph $K_{m,n}$ is a graph comprised of two partite sets of vertices of sizes m and n , respectively, such that two vertices are adjacent if and only if they are in different partite sets.

Let $G = (V, E)$ be a graph. The *neighborhood* of a vertex u in a subgraph $H \subseteq G$, denoted by $N_H(u)$, is the set of all vertices adjacent to u in H . The degree of a vertex u in G , denoted by $d_G(u)$, is the number of the vertices adjacent to u in G . Note that $d_G(u) = |N_G(u)|$. For a vertex subset $V' \subseteq V$, the neighborhood of V' in a subgraph $H \subseteq G$ is defined as $N_H(V') = \bigcup_{v \in V'} (N_H(v)) - V'$.

A *group* is a nonempty set G together an binary operation $*$ defined on G and satisfies the following properties:

- (1) $*$ is associative, that means $(a * b) * c = a * (b * c)$ for any $a, b, c \in G$;
- (2) $(G, *)$ has an identity e , that means $a * e = e * a = a$ for any $a \in G$;
- (3) For any element $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$;

Throughout this paper, all groups are finite. If a subset H of a group G is itself a group under the operation of G , we say that H is a *subgroup* of G and denoted by $H \leq G$. A subgroup H of a group G is called *normal*, denoted by $H \trianglelefteq G$, if $H^g = H$, $\forall g \in G$. For a subset S of a group G , the intersection of all subgroups of G containing S is called the *subgroup generated by S* , denoted by $\langle S \rangle$. For an element g in a group G , the *order* of g is the smallest positive integer n satisfying $g^n = e$. An element of G of order 2 is also called an *involution*. Clearly, if g is an involution, then $g = g^{-1}$.

Let n be a positive integer. Throughout this paper, \mathbb{Z}_n represents the cyclic group of order n as well as the ring of integers modulo n .

An *isomorphism* from a simple graph G to a simple graph H is a bijection $\pi : V(G) \rightarrow V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\pi(u), \pi(v)\} \in E(H)$. If there is an isomorphism from G to H , we say that G and H are *isomorphic* and write $G \cong H$. An isomorphism from the graph G onto itself is called an *automorphism* of G . The set of all automorphisms of the graph G , with the operation of composition, is the automorphism group of G , denoted by $\text{Aut}(G)$. We say that G is *vertex-transitive* if for any two vertices $u, v \in V(G)$, there exists an automorphism $\pi \in \text{Aut}(G)$ such that $\pi(u) = v$.

Given a finite group G and a subset $S \subseteq G \setminus \{e\}$ such that $S = S^{-1} = \{s^{-1} \mid s \in S\}$, where e is the identity element of G , the *Cayley graph* $\text{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A Cayley graph $\text{Cay}(G, S)$ is connected if and only if S generates G . Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G , is a permutation group isomorphic to G . It is well-known that $R(G) \leq \text{Aut}(\text{Cay}(G, S))$. So, $\text{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G , acting regularly on the vertex set of X (see [1, Lemma 16.3]).

2.2 Preliminary results

Given a positive integer n , let $f_n(g) = n(g+1) - \frac{g(g+3)}{2}$ be a function of g .

Proposition 2.1 [28, Remark 2.2] or [30, Lemma 3.1] *If $0 \leq g \leq n-2$, then $f_n(g)$ is strictly monotonically increasing. Moreover, the maximum of $f_n(g)$ is $f_n(n-2) = \frac{n(n-1)}{2} + 1$ and $f_n(n-1) = f_n(n-2) > f_n(n) = f_n(n-3) > f_n(g)$ for $0 \leq g \leq n-4$.*

The following result first appeared in [28, Remark 2.2] without proof. Here we give a detailed proof.

Lemma 2.2 *Let $0 \leq g_1, g_2 \leq n-2$ and $0 \leq g \leq n-3$. If $g_1 + 1 + g_2 + 1 > g + 1$, then $f_{n-1}(g_1) + f_{n-1}(g_2) \geq f_n(g) + 1$.*

Proof We first assume that $g_1, g_2 > g$. By Proposition 2.1,

$$f_{n-1}(g_1) + f_{n-1}(g_2) \geq 2f_{n-1}(g) \geq f_n(g) + f_{n-2}(0).$$

Since $n \geq 3$, one has $f_{n-2}(0) = n-2 \geq 1$, and so $f_{n-1}(g_1) + f_{n-1}(g_2) \geq f_n(g) + 1$, as required.

Now assume that either $g_1 \leq g$ or $g_2 \leq g$. Without loss of generality, let $g_1 \leq g$. Then $g_2 > g - g_1 - 1$. Clearly, $g - g_1 - 1 \leq g - 1 \leq (n-1) - 3$.

If $g - g_1 - 1 < 0$, then $g_1 > g - 1$, and since $g_1 \leq g$, one has $g_1 = g$. It follows that

$$f_{n-1}(g_1) + f_{n-1}(g_2) \geq f_{n-1}(g) + f_{n-1}(0) = f_n(g) + (n - g - 2).$$

As $g \leq n-3$, one has $n - g - 2 \geq 1$, implying $f_{n-1}(g_1) + f_{n-1}(g_2) \geq f_n(g) + 1$, as required.

If $g - g_1 - 1 \geq 0$, then by Proposition 2.1, we have

$$\begin{aligned} f_{n-1}(g_1) + f_{n-1}(g_2) &> f_{n-1}(g_1) + f_{n-1}(g - g_1 - 1) \\ &= f_n(g) + g_1(g - g_1 - 1) \geq f_n(g). \end{aligned}$$

It follows that $f_{n-1}(g_1) + f_{n-1}(g_2) \geq f_n(g) + 1$, as required. \square

The following results are very useful when used to study the extra connectivity of HL-networks.

Proposition 2.3 [15, Lemma 4] *For any integer $g \geq 0$ and any integer $n \geq \lceil \frac{g+2}{2} \rceil$, for any $X_n \in \mathbb{L}_n$ and $U \subseteq V(X_n)$ with $|U| = g+1$, we have $|N_{X_n}(U)| \geq f_n(g)$.*

Proposition 2.4 [29, Lemma 4 & Corollary 1] *Let $X_n \in \mathbb{L}_n$. Then the girth of X_n is 4, and any two vertices of X_n have at most two common neighbors.*

The following result is about the hypercubes.

Proposition 2.5 [28, Lemma 2.1] *For any integer $g \geq 0$ and any integer $n \geq 4$, for any $U \subseteq V(Q_n)$ with $|U| = g + 1$, we have $|N_{Q_n}(U)| \geq f_n(g)$.*

Following Latifi [19], we express Q_n as $D_0 \odot D_1$, where D_0 and D_1 are the two $(n-1)$ -subcubes of Q_n induced by the vertices with the i th coordinates 0 and 1, respectively. Sometimes we use $X^{i-1}0X^{n-i}$ and $X^{i-1}1X^{n-i}$ to denote D_0 and D_1 , where $X \in \mathbb{Z}_2$. Clearly, the vertex v in one $(n-1)$ -subcube has exactly one neighbor v_0 in another $(n-1)$ -subcube. The following lemma presents a generalization of [28, Theorem 3.2]

Lemma 2.6 *Let $n \geq 4$ and $g \geq 0$. Let $U \subseteq V(Q_n)$ such that $|U| = g + 1$ and $Q_n[U]$ is connected. If $|N_{Q_n}(U)| = f_n(g)$, then $Q_n[U]$ is a star.*

Proof We will verify the lemma by induction on g . The result is clearly true for $g = 0$. We assume that $g \geq 1$ and the result is true for $h < g$. Next, we verify that this result is also true for $h = g$. Since $|U| \geq 2$, we can take two distinct vertices, say $x = x_1x_2 \dots x_i \dots x_n$ and $y = y_1y_2 \dots y_i \dots y_n$ in U such that $x_i = 0$ and $y_i = 1$. Let $V_0 = V(X^{i-1}0X^{n-i})$ and $V_1 = V(X^{i-1}1X^{n-i})$. Then $x \in V_0$ and $y \in V_1$, and so $U_i = V_i \cap U$ is non-empty for $i = 0, 1$. Without loss of generality, assume that $|U_0| \leq |U_1|$. Letting $|U_0| = N$, we have $N \leq \lfloor \frac{g+1}{2} \rfloor$. By Proposition 2.5, $|N_{V_0}(U_0)| \geq f_{n-1}(N-1)$ and $|N_{V_1}(U_1)| \geq f_{n-1}(g-N)$. Note that $N_{V_0}(U_0) \cup N_{V_1}(U_1) \subseteq N_{Q_n}(U)$ and $N_{V_0}(U_0) \cap N_{V_1}(U_1) = \emptyset$. It follows that $|N_{V_0}(U_0)| + |N_{V_1}(U_1)| \leq |N_{Q_n}(U)| = f_n(g)$, and hence

$$\begin{aligned} 0 &\geq |N_{V_0}(U_0)| + |N_{V_1}(U_1)| - |N_{Q_n}(U)| \\ &\geq f_{n-1}(N-1) + f_{n-1}(g-N) - f_n(g) \\ &= -(N-1)(N-g). \end{aligned} \tag{1}$$

Remember that $1 \leq N \leq \lfloor \frac{g+1}{2} \rfloor$. If $N > 1$, then $N < g$ and so $0 \geq -(N-1)(N-g) > 0$, a contradiction. Thus, $N = 1$ and so $|U_1| = g$. Let $U_0 = U \cap V_0 = \{v\}$. Since v has only one neighbor in V_1 , v is a pendant vertex of $Q_n[U]$, and so $Q_n[U_1]$ is connected. Since $N = 1$, by the above equation (1), we have $|N_{V_1}(U_1)| = f_{n-1}(g-1)$ and $|N_{Q_n}(U)| = |N_{V_0}(U_0)| + |N_{V_1}(U_1)|$. It follows that $|N_{Q_n}(U_1)| = |N_{V_1}(U_1)| + |N_{V_0}(U_1)| = f_{n-1}(g-1) + g = f_n(g-1)$. By the induction hypothesis, we have $Q_n[U_1]$ is a star. Assume that $U_1 = \{u, u_1, \dots, u_{g-1}\}$ and $E(Q_n[U_1]) = \{\{u, u_i\} \mid 1 \leq i \leq g-1\}$. If v is adjacent to u , then $Q_n[U] \cong K_{1,g}$. Assume that v is adjacent to some u_i . If $g \leq 2$, then $Q_n[U]$ must be a star. Suppose $g > 2$. If there is a $w \in N_{V_0}(U_1 - u_i) \setminus N_{V_0}(v)$, then $|N_{V_0}(U_0)| + |N_{V_1}(U_1)| = |N_{Q_n}(U)| \geq 1 + |N_{V_0}(U_0)| + |N_{V_1}(U_1)|$, a contradiction. Thus, $N_{V_0}(U_1 - u_i) \subseteq N_{V_0}(v)$. Take $u_j \in U_1 - \{u, u_i\}$. Then the neighbor w_j of u_j in V_0 is adjacent to v . This implies that (v, u_i, u, u_j, w_j) is a cycle of length 5, contrary to the fact that Q_n is bipartite. \square

3 Lower bound on g -extra connectivity of HL-networks

A graph is said to be *hyper- κ_g* if the deletion of each minimum R_g -cutset creates exactly two components, one of which has $g + 1$ vertices. Clearly, a hyper- κ_0 graph is also hyper- κ

(see [20] for the definition of hyper- κ). The following lemma shows that every HL-network is hyper- κ .

Lemma 3.1 *For any $X_n \in \mathbb{L}_n$, X_n is hyper- κ for $n \geq 2$.*

Proof We will prove the lemma by induction on n . Let S be a minimum vertex-cut of X_n . By [14], we have $|S| = n$. The result is clearly true when $n = 2$. In what follows, assume that $n \geq 3$, and that the result holds for X_{n-1} .

Suppose $X_n = X_{n-1}^0 \oplus X_{n-1}^1$. Let $S_i = S \cap V(X_{n-1}^i)$ with $i = 0, 1$. Since $n \geq 3$, one has $2^{n-1} - n \geq 1$, and so there is at least one edge between $X_{n-1}^0 - S_0$ and $X_{n-1}^1 - S_1$. Without loss of generality, assume that $|S_0| \leq |S_1|$. If $|S_0| = 0$, then X_{n-1}^0 is connected, and since each vertex of X_{n-1}^1 has a neighbor in X_{n-1}^0 , it follows that $X_n - S$ is connected, a contradiction. Thus, $|S_0| \geq 1$. If $|S_0| > 1$, then $|S_0| \leq |S_1| < n - 1$, and so $X_{n-1}^i - S_i$ is connected for $i = 0, 1$. Since there is at least one edge between $X_{n-1}^0 - S_0$ and $X_{n-1}^1 - S_1$, $X_n - S$ is still connected, a contradiction. Consequently, $|S_0| = 1$, $|S_1| = n - 1$ and $X_{n-1}^1 - S_1$ is disconnected. By the induction hypothesis, $X_{n-1}^1 - S_1$ has exactly two components, one of which is a singleton, say u . Clearly, $X_{n-1}^0 - S_0$ is connected. If $n \geq 4$, then $2^{n-1} - n \geq 4$, and so there are at least four edges between $X_{n-1}^0 - S_0$ and $X_{n-1}^1 - S_1$, the component of $X_{n-1}^1 - S_1$ which is not the singleton is connected to $X_{n-1}^0 - S_0$. It follows that $X_n - S$ has exactly two components, one of which is the singleton u , as required. If $n = 3$, then $X_{n-1}^1 - S_1$ is a null graph with two vertices, say u, v . Since there is at least one edge between $X_{n-1}^0 - S_0$ and $X_{n-1}^1 - S_1$, either u or v is connected to $X_{n-1}^0 - S_0$. Again, $X_n - S$ has exactly two components, one of which is a singleton, as required. \square

As a slight generalization of [30, Theorem 3.3], the following result shows that an n -dimensional HL-network with at most $f_n(g)$ faulty vertices, where $0 \leq g \leq n - 3$, has a very large component.

Lemma 3.2 *Let $n \geq 5$, $0 \leq g \leq n - 3$. For any $X_n \in \mathbb{L}_n$, let $S \subseteq V(X_n)$. If $|S| \leq f_n(g) - k$ with $0 \leq k \leq 1$, then $X_n - S$ has a component with at least $2^n - |S| - (g + 1 - k)$ vertices.*

Proof We will prove the lemma by induction on g . Let $g = 0$. Then $|S| \leq n - k$. By Lemma 3.1, if $k = 1$, then $G - S$ is connected, and if $k = 0$ and $|S| = n$, then $G - S$ either is connected, or has two components, one of which is a singleton. This implies that the result is true for $g = 0$.

In what follows, assume that $g \geq 1$, and that the result holds for $g - 1$. We shall verify that it is also true for g . Suppose $X_n = X_{n-1}^0 \oplus X_{n-1}^1$. Let $S_i = S \cap V(X_{n-1}^i)$ with $i = 0, 1$. Without loss of generality, assume that $|S_0| \leq |S_1|$.

Case 1 $|S_0| \leq n - 2$

In this case, $X_{n-1}^0 - S_0$ is connected because $\kappa(X_{n-1}^0) = n - 1$ by Lemma 3.1. Let C be the component of $X_n - S$ containing $X_{n-1}^0 - S_0$, and let C' be the union of all other components of $X_n - S$. Then $C' \subseteq X_{n-1}^1$. Noting that the edges between $V(X_{n-1}^0)$

and $V(X_{n-1}^1)$ form a perfect matching, one has $|V(C')| = |N_{X_{n-1}^0}(V(C'))| \leq |S_0| \leq n-2$. Clearly, $|N_{X_n}(V(C'))| \leq |S|$. By Proposition 2.3, we have $|N_{X_n}(V(C'))| \geq f_n(|V(C')| - 1)$. It follows that $f_n(g) - k \geq |S| \geq |N_{X_n}(V(C'))| \geq f_n(|V(C')| - 1)$. By Proposition 2.1, if $k = 0$, then $|V(C')| - 1 \leq g$ and so $|V(C')| \leq g + 1$; if $k = 1$, then $|V(C')| - 1 < g$ and so $|V(C')| \leq g = (g + 1) - 1$. Thus, we always have $|V(C')| \leq (g + 1) - k$ and so $|V(C)| \geq 2^n - |S| - (g + 1 - k)$.

Case 2 $|S_0| > n - 2$

In this case, we have

$$\begin{aligned} |S_0| \leq |S_1| &= |S| - |S_0| \\ &\leq f_n(g) - k - (n - 1) \\ &= f_{n-1}(g - 1) - k. \end{aligned}$$

By the induction hypothesis, for each $i = 0, 1$, X_{n-1}^i has a component A_i with at least $2^{n-1} - |S_i| - (g - k)$ vertices. Clearly, $|S| + 2g \leq f_n(g) - k + 2g$. By Proposition 2.1, $f_n(g) \leq f_n(n-3) = \frac{n^2-n}{2}$. It follows that $|S| + 2g \leq f_n(n-3) - k + 2(n-3) = \frac{n^2+3n-12-2k}{2} < 2^{n-1}$ because $n \geq 5$. In view of the fact that there are 2^{n-1} vertex disjoint edges between $V(X_{n-1}^0)$ and $V(X_{n-1}^1)$, there exist some edges between A_0 and A_1 in $X_n - F$. Let C be the component of $X_n - S$ containing A_1 and A_2 , and let C' be the union of all other components of $X_n - S$. Let $V_i = V(C') \cap V(X_{n-1}^i)$ for $i = 0, 1$. Then $|V_0|, |V_1| \leq g - k \leq n - 3$. Note that $|S| \geq |N_{X_{n-1}^0}(V_0)| + |N_{X_{n-1}^1}(V_1)|$. By Proposition 2.3, $|N_{X_{n-1}^0}(V_0)| + |N_{X_{n-1}^1}(V_1)| \geq f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1)$. It follows that $|S| \geq f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1)$. Since $|S| \leq f_n(g) - k$, one has $f_n(g) - k \geq f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1)$. From Lemma 2.2 it follows that $|V_0| + |V_1| \leq g + 1$. If $k = 0$, then $|V_0| + |V_1| \leq g + 1 = g + 1 - k$. Let $k = 1$. Suppose that $|V_0| + |V_1| = g + 1$. Letting $g_0 = |V_0| - 1$, we have

$$f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1) = f_n(g) + g_0[g - g_0 - 1].$$

As $g - g_0 - 1 = |V_1| - 1 \geq 0$, one has $f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1) \geq f_n(g)$, contrary to the fact that $f_n(g) - 1 \geq f_{n-1}(|V_0| - 1) + f_{n-1}(|V_1| - 1)$. Thus, $|V_0| + |V_1| \leq g$. As a result, we always have $|V(C')| = |V_0| + |V_1| \leq g + 1 - k$ and so $|V(C)| \geq 2^n - |S| - (g + 1 - k)$, as required. \square

By Lemma 3.2, we immediately have the following result.

Theorem 3.3 *Let $n \geq 5$, $0 \leq g \leq n - 3$. For any $X_n \in \mathbb{L}_n$, let S be a R_g -cutset of X_n . If $|S| \leq f_n(g) = n(g + 1) - \frac{g(g+3)}{2}$, then $|S| = f_n(g)$ and X_n is hyper- κ_g . In particular, $\kappa_g(X_n) \geq f_n(g)$.*

Proof If $|S| \leq f_n(g) - 1$ then by Lemma 3.2, $X_n - S$ would have a component with at most g vertices, contrary to the fact that S is a R_g -cutset. Therefore, $|S| = f_n(g)$. Again, by Lemma 3.2, $X_n - S$ has a component, say C , with at least $2^n - |S| - (g + 1)$ vertices. Let A be a component of $X_n - S$ different from S . Then $|V(A)| \leq g + 1$. Since S is a R_g -cutset, we must have $|V(A)| = g + 1$. Therefore, $X_n - S$ has exactly two components that are C and A . \square

Remark 1 After this work was finished, it came to our notice that Yang and Liu [27, Remark 4.11] also proved that for $0 \leq g \leq n-4$, if X_n has a R_g -cutset S with $|S| \leq f_n(g)$, then $\kappa_g(X_n) = f_n(g)$. With our approach, however, we were able to further obtain that X_n is also hyper- κ_g .

4 Some HL-networks X_n with $\kappa_g(X_n) = f_n(g)$

In this section, we shall show that the lower bound on $\kappa_g(X_n)$ in Theorem 3.3 is best possible. We first give a sufficient condition for the equality $\kappa_g(X_n) = f_n(g)$.

Theorem 4.1 *Let $n \geq 5$, $0 \leq g \leq n-3$, and $X_n \in \mathbb{L}_n$. If X_n has a connected subgraph, say A , such that $|V(A)| = g+1$ and $|N_{X_n}(A)| = f_n(g)$, then $\kappa_g(X_n) = f_n(g)$.*

Proof By Theorem 3.3, it suffices to show that $\kappa_g(X_n) \leq f_n(g)$. Set $S = N_{X_n}(A)$. Then A is a component of $G - S$ with $g+1$ vertices. Since $|S| = f_n(g)$, from Lemma 3.2 it follows that $G - S$ has a component, say A' , with at least $2^n - |S| - (g+1)$ vertices. By Proposition 2.1, $2^n - f_n(g) - (g+1) \geq 2^n - f_n(n-3) - (n-2)$. Since $n \geq 5$, one has $2^n - f_n(n-3) - (n-2) = 2^n - \frac{n^2+n-4}{2} > n \geq g+3$. It follows that $|V(A')| \geq 2^n - |S| - (g+1) = 2^n - f_n(g) - (g+1) > g+3$, and so $A' \neq A$. Then $V(X_n) = V(A) \cup V(A') \cup S$, which implies that S is a g -extra vertex cut of X_n . Hence, $\kappa_g(X_n) \leq f_n(g)$. \square

This theorem is very powerful when used to determine the g -extra connectivity of HL-networks for small g . For example, we can use it to give a short proof of [4, Theorems 1,2].

Corollary 4.2 *For any $X_n \in \mathbb{L}_n$, $\kappa_2(X_n) = 3n-5$ for $n \geq 5$ and $\kappa_3(X_n) = 4n-9$ for $n \geq 6$.*

Proof From the definition of HL-networks, X_n has a subgraph isomorphic to a k -dimensional HL-network for each $1 \leq k \leq n$. Pick a subgraph, say C , isomorphic to a 2-dimensional HL-network. Then C is a 4-cycle. Let $C = (u_0, u_1, u_2, u_3, u_0)$ and let $A = u_0 u_1 u_2$. By Proposition 2.4, any two vertices of X_n have at most two common neighbors. Then $|N_{X_n}(A)| = 3n-5 = f_n(2)$. Since $n \geq 5$, one has $g = 2 \leq n-3$. From Theorem 4.1 it follows that $\kappa_2(X_n) = 3n-5$.

Now take a subgraph, say H , isomorphic to a 3-dimensional HL-network. Then H is one of the following two graphs in Figure 2. If H is the first graph, then let A be the

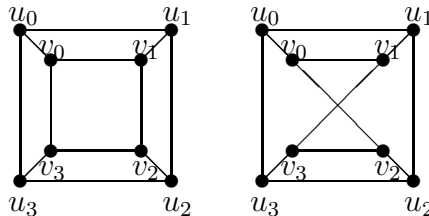


Figure 2: Two 3-dimensional HL-networks

star $K_{1,3}$ with vertices u_0, v_0, u_1, u_3 . If H is the second graph, then let A be the 3-path $u_0v_0v_1v_3$. For both cases, it is also easy to check that $|N_{X_n}(A)| = 4n - 9 = f_n(3)$. Since $n \geq 6$, one has $g = 3 \leq n - 3$. By Theorem 4.1, we have $\kappa_3(X_n) = 4n - 9$. \square

In what follows, we shall consider the g -extra connectivity of the varietal hypercubes which were proposed by Cheng and Chuang [7]. An n -dimensional varietal hypercube, denoted by VQ_n , is defined recursively as follows.

Definition 4.3 *The VQ_1 is the complete graph of two vertices labeled with 0 and 1, respectively. Assume that VQ_{n-1} has been constructed for $n \geq 1$. Let VQ_n^0 (resp. VQ_n^1) be the graph obtained from VQ_{n-1} by inserting a 0 (resp. 1) in front of each vertex-labeling in VQ_{n-1} . Then the VQ_n is obtained by joining vertices in VQ_n^0 and VQ_n^1 , according to the rule: a vertex $0x_{n-1}x_{n-2}x_{n-3} \dots x_2x_1$ in VQ_n^0 and a vertex $0y_{n-1}y_{n-2}y_{n-3} \dots y_2y_1$ in VQ_n^1 are adjacent in VQ_n if and only if one of the following holds:*

- (1) $x_{n-1}x_{n-2}x_{n-3} \dots x_2x_1 = y_{n-1}y_{n-2}y_{n-3} \dots y_2y_1$ if $3 \mid n$;
- (2) $x_{n-3} \dots x_2x_1 = y_{n-3} \dots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(00, 00), (01, 01), (10, 11), (11, 10)\}$ if $3 \nmid n$.

Clearly, for $n \geq 1$, VQ_n is an n -regular graph with vertex set $V = \{x_nx_{n-1} \dots x_2x_1 \mid x_i \in \mathbb{Z}_2, 1 \leq i \leq n\}$. In what follows, we always assume that $n = 3s + t$, where $0 \leq t \leq 2$ and s is a non-negative integer.

Lemma 4.4 *Let $u = x_nx_{n-1} \dots x_3x_2x_1 \in V(VQ_n)$. Then a vertex $v = y_ny_{n-1} \dots y_3y_2y_1$ is adjacent to u if and only if one of the following holds:*

- (1) for some $1 \leq i \leq n$ with $3 \nmid i$, $y_i = \overline{x_i}$, and for all $1 \leq j \leq n$ with $j \neq i$, $y_j = x_j$;
- (2) for some $1 \leq i \leq n$ with $3 \mid i$, $y_i = \overline{x_i}$, $y_{i-2} = x_{i-1} + x_{i-2}$, and for all $1 \leq j \leq n$ with $j \neq i, i-2$, $y_j = x_j$.

Proof We shall first prove the sufficiency by using induction on n . The result is obviously true for $n = 1$ because VQ_1 is a complete graph with two vertices $\{0, 1\}$. Now assume that the result holds for $n - 1$. If $i < n$, then by induction, $x_{n-1} \dots x_3x_2x_1$ is adjacent to $y_{n-1} \dots y_3y_2y_1$ in VQ_{n-1} . Since $x_n = y_n$, either $u, v \in V(VQ_n^0)$ or $u, v \in V(VQ_n^1)$. It follows that u and v are also adjacent in VQ_n . If $i = n$, we may assume that $x_n = 0$ and $y_n = 1$. If $3 \nmid n$, then $y_j = x_j$ for all $0 \leq j \leq n - 1$, and then by the definition of VQ_n , u and v are adjacent. If $3 \mid n$, then $y_{n-2} = x_{n-1} + x_{n-2}$, and $y_j = x_j$ for all $1 \leq j \leq n$ with $j \neq n, n - 2$. Noting that $x_i \in \mathbb{Z}_2$ for all $1 \leq i \leq n$, it is easy to check that $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(00, 00), (01, 01), (10, 11), (11, 10)\}$. By the definition of VQ_n , u and v are adjacent.

For the necessity, let N be the set of vertices whose coordinates satisfy the condition (1) or (2). By the sufficiency, each vertex in N is adjacent to u . Clearly, $|N| = n$, so N is just the neighborhood of u in VQ_n because VQ_n has valency n . So, $v \in N$. \square

Remember that $n = 3s + t$ with $s \geq 0$ and $0 \leq t \leq 2$. For $1 \leq i \leq s$, let H_i be the dihedral group of order 8 defined as follows:

$$H_i = \langle a_i, b_i \mid a_i^4 = b_i^2 = 1, b_i^{-1}a_i b_i = a_i^{-1} \rangle.$$

Let $\langle c_1 \rangle \times \dots \times \langle c_t \rangle \cong \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{t \text{ times}}$. Set $G_n = H_1 \times H_2 \times \dots \times H_s \times \langle c_1 \rangle \times \dots \times \langle c_t \rangle$, and $\Omega_n = \{a_i^2, b_i, a_i b_i, c_1, \dots, c_t \mid 1 \leq i \leq s\}$. Let $\Delta_n = \text{Cay}(G_n, \Omega_n)$.

Theorem 4.5 $\Delta_n \cong VQ_n$.

Proof We define a map f from $V(\Delta_n)$ to $V(VQ_n)$ as follows:

$$\begin{aligned} & (a_1^2)^{x_1} b_1^{x_2} (a_1 b_1)^{x_3} \dots (a_s^2)^{x_{3s-2}} b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}} \\ & \mapsto x_{3s+t} \dots x_{3s+1} x_{3s} x_{3s-1} x_{3s-2} \dots x_3 x_2 x_1, \end{aligned}$$

where $x_i \in \mathbb{Z}_2$ with $1 \leq i \leq n$. It is easy to check that f is a bijection. Take an edge, say $\{u, v\}$, of Δ_n . Without loss of generality, let

$$u = (a_1^2)^{x_1} b_1^{x_2} (a_1 b_1)^{x_3} \dots (a_s^2)^{x_{3s-2}} b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}}.$$

Then $v = gu$ for some $g \in \Omega_n$.

If $g = c_i$ for some $0 \leq i \leq t$, then

$$v = (a_1^2)^{x_1} b_1^{x_2} (a_1 b_1)^{x_3} \dots (a_s^2)^{x_{3s-2}} b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_i^{x_{3s+i}+1} \dots c_t^{x_{3s+t}},$$

and so $v^f = x_{3s+t} \dots \overline{x_{3s+i}} \dots x_{3s+1} x_{3s} x_{3s-1} x_{3s-2} \dots x_3 x_2 x_1$. By Lemma 4.4, $\{u^f, v^f\} \in E(VQ_n)$.

If $g = a_i^2$ for some $1 \leq i \leq s$, then

$$v = (a_1^2)^{x_1} \dots (a_i^2)^{x_{3i-2}+1} (b_i)^{x_{3i-1}} (a_i b_i)^{x_{3i}} \dots b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}},$$

and so $v^f = x_{3s+t} \dots x_{3s+1} x_{3s} \dots x_{3i} x_{3i-1} \overline{x_{3i-2}} \dots x_2 x_1$. By Lemma 4.4, $\{u^f, v^f\} \in E(VQ_n)$.

If $g = b_i$ for some $1 \leq i \leq s$, then

$$v = (a_1^2)^{x_1} \dots (a_i^2)^{x_{3i-2}} (b_i)^{x_{3i-1}+1} (a_i b_i)^{x_{3i}} \dots b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}},$$

and so $v^f = x_{3s+t} \dots x_{3s+1} x_{3s} \dots x_{3i} \overline{x_{3i-1}} x_{3i-2} \dots x_2 x_1$. Again, by Lemma 4.4, $\{u^f, v^f\} \in E(VQ_n)$.

If $g = a_i b_i$ for some $1 \leq i \leq s$, then

$$\begin{aligned} v &= g(a_1^2)^{x_1} \dots (a_i^2)^{x_{3i-2}} (b_i)^{x_{3i-1}} (a_i b_i)^{x_{3i}} \dots b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}} \\ &= (a_1^2)^{x_1} \dots (a_i^2)^{x_{3i-2}} g(b_i)^{x_{3i-1}} (a_i b_i)^{x_{3i}} \dots b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}} \\ &= (a_1^2)^{x_1} \dots (a_i^2)^{x_{3i-2}+x_{3i-1}} (b_i)^{x_{3i-1}} (a_i b_i)^{x_{3i}+1} \dots b_s^{x_{3s-1}} (a_s b_s)^{x_{3s}} c_1^{x_{3s+1}} \dots c_t^{x_{3s+t}}. \end{aligned}$$

and so $v^f = x_{3s+t} \dots x_{3s+1} x_{3s} \dots \overline{x_{3i}} x_{3i-1} (x_{3i-1} + x_{3i-2}) \dots x_2 x_1$. Again, by Lemma 4.4, $\{u^f, v^f\} \in E(VQ_n)$.

Now we see that f is an isomorphism from Δ_n to VQ_n . Therefore, $VQ_n \cong \Delta_n$. \square

From Theorem 4.5, we see that VQ_n is a Cayley graph, and so it is vertex-transitive.

Corollary 4.6 [24, Theorem 2.5] VQ_n is vertex-transitive.

Theorem 4.7 *Let $n = 3s + t$ with $s \geq 3$ and $0 \leq t \leq 2$. If $0 \leq g \leq n - s$, then $\kappa_g(VQ_n) = f_n(g)$.*

Proof By Theorem 4.5, $VQ_n \cong \Delta_n$, and so $\kappa_g(VQ_n) = \kappa_g(\Delta_n)$. Set $\Omega' = \{a_i^2, b_i, c_1, \dots, c_t \mid 1 \leq i \leq s\}$. Clearly, $|\Omega'| = n - s$. Since $0 \leq g \leq n - s$, we can take a subset V' of $V(\Delta_n)$ such that $|V'| = g + 1$, $e \in V'$ and $V' - \{e\} \subset \Omega'$, where e is the identity element of the group G_n . Clearly, $\Delta_n[V'] \cong K_{1,g}$. Note that the elements in Ω' are pair-wise commutative. So, for any distinct $a, a' \in \Omega'$, e and aa' are two distinct common neighbors of a and a' . From Proposition 2.4 it follows that any two elements in Ω' have exactly two common neighbors in Δ_n . With an easy calculation, we see that $|N_{\Delta_n}(V')| = f_n(g)$. By Theorem 4.1, we have $\kappa_g(\Delta_n) = f_n(g)$. \square

5 A negative answer to Problem A

For an n -dimensional HL-network X_n , if $0 \leq g \leq n - 3$ and $n \geq 5$, then we have $\kappa_g(X_n) \geq f_n(g)$ by Theorem 3.3. In this section, we shall construct a class of HL-networks with g -extra connectivity greater than $f_n(g)$.

Let k, ℓ be two non-negative integers. Let $\langle c_1 \rangle \times \dots \times \langle c_\ell \rangle \cong \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{\ell \text{ times}}$, and for $1 \leq i \leq k$, let H_i be the dihedral group of order 8 defined as follows:

$$H_i = \langle a_i, b_i \mid a_i^4 = b_i^2 = 1, b_i^{-1}a_ib_i = a_i^{-1} \rangle.$$

Set $G_{k,\ell} = H_1 \times H_2 \times \dots \times H_k \times \langle c_1 \rangle \times \dots \times \langle c_\ell \rangle$. Let $\Gamma_{k,\ell}$ be the Cayley graph $\text{Cay}(G_{k,\ell}, \Omega_{k,\ell})$, where $\Omega_{k,\ell} = \{a_i^2, b_i, a_ib_i, c_1, \dots, c_\ell \mid 1 \leq i \leq k\}$. Let $\mathbb{G}_n = \{G_{k,\ell} \mid 3k + \ell = n\}$ and let $\mathcal{G}_n = \{\Gamma_{k,\ell} \mid 3k + \ell = n\}$.

Lemma 5.1 *Each graph in \mathcal{G}_n is isomorphic to an n -dimensional HL-network.*

Proof We shall verify the result by using induction on n . If $n = 1$, then $k = 0$ and $\ell = 1$, and so $\mathcal{G}_1 = \{\Gamma_{0,1}\}$. Clearly, $\Gamma_{0,1} = \text{Cay}(\langle c_1 \rangle, \{c_1\}) \cong K_2$, which belongs to \mathbb{L}_1 . In what follows, assume that $n > 1$ and that the result is true for $h \leq n - 1$. We will verify that it is also true for $h = n$.

Let $k = 0$. Then $\mathcal{G}_n = \mathcal{G}_\ell = \{\Gamma_{0,\ell}\}$. Note that $G_{0,\ell} \cong \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{\ell \text{ times}}$, and $\Omega_{0,\ell} = \{c_1, \dots, c_\ell\}$. It is easy to check that $\Gamma_{0,\ell} = \text{Cay}(G_{0,\ell}, \Omega_{0,\ell}) \cong Q_\ell$ which belongs to \mathbb{L}_ℓ .

Let $k > 0$. Take any $\Gamma_{k,\ell} \in \mathbb{G}_n$. Then $\Gamma_{k,\ell} = \text{Cay}(G_{k,\ell}, \Omega_{k,\ell})$. Set $M = H_1 \times \dots \times H_{k-1} \times \langle a_k^2 \rangle \times \langle a_kb_k \rangle \times \langle c_1 \rangle \times \dots \times \langle c_\ell \rangle$. Clearly, M is a subgroup of $G_{k,\ell}$ of index 2, and $M \cong G_{k-1,\ell+2}$. Set $A = \text{Cay}(M, \Omega_{k,\ell} - \{b_k\})$. It is easy to see that $A \cong \Gamma_{k-1,\ell+2} \in \mathcal{G}_{n-1}$. By the induction hypothesis, A is isomorphic to an $(n-1)$ -dimensional HL-network. Note that M has two cosets in $G_{k,\ell}$ that are M and Mb_k . In view of the fact that the map $R(b_k) : v \mapsto vb_k, \forall v \in G_{k,\ell}$ is an automorphism of $\Gamma_{k,\ell}$, it follows that $\Gamma_{k,\ell}[Mb_k] \cong A$, and

hence it is also isomorphic to an $(n - 1)$ -dimensional HL-network. By the definition of Cayley graph, we see that for any $u \in M$, $b_k u$ is the unique neighbor of u in Mb_k . This implies that the edges between M and Mb_k form a perfect matching. Therefore, $\Gamma_{k,\ell}$ is isomorphic to an n -dimensional HL-network. \square

Lemma 5.2 *For any $\Gamma_{k,\ell} \in \mathcal{G}_n$, and for any $u, w \in N_{\Gamma_{k,\ell}}(v)$, if u and w have a unique common neighbor, say v , then $\{u, w\} = \{b_i v, a_i b_i v\}$ for some $1 \leq i \leq k$.*

Proof Since v is a common neighbor of u and w , by the definition of Cayley graph, we have $u = gv$ and $w = g'v$ for some $g, g' \in \Omega_{k,\ell}$. If g commutes with g' , then $gg'v = g'gv$ is also a common neighbor of u and w . The uniqueness of v implies that $gg'v = v$, and so $gg' = e$, where e is the identity element of the group $G_{k,\ell}$. Note that all elements of $\Omega_{k,\ell}$ are involutions (elements of order 2). It follows that $g = g'$, forcing that $u = gv = g'v = w$, a contradiction. Thus, g does not commute with g' . By the structure of the group $G_{k,\ell}$, we see that the only possibility is $\{g, g'\} = \{b_i, a_i b_i\}$ for some $1 \leq i \leq k$, and so $\{u, w\} = \{b_i v, a_i b_i v\}$. \square

Lemma 5.3 *Let $n \geq 5$ and let $0 \leq g \leq n - 4$. For any $\Gamma_{k,\ell} \in \mathcal{G}_n$, let A be a connected subgraph of $\Gamma_{k,\ell}$ such that $|V(A)| = g + 1$. If $|N_{\Gamma_{k,\ell}}(V(A))| = f_n(g)$, then A is isomorphic either to $K_{1,g}$ or to P_3 .*

Proof We shall prove the theorem by using induction on g . Obviously, the result is true for $g = 0$. In what follows, assume that $g > 0$ and the result is true for $h < g$. We shall verify that it is also true for $h = g$.

By the vertex-transitivity of $\Gamma_{k,\ell}$, we may assume that the identity element e of the group $G_{k,\ell}$ is contained in $V(A)$. We consider the following two cases:

Case 1 For any edge $\{x, y\}$ of A , $xy^{-1} \in \{a_i^2 \mid 1 \leq i \leq k\}$.

For any vertex, say v , of A , A has a path with end vertices e and v because of the connectivity of A . Assume that this path is $v_0 v_1 v_2 \dots v_j$ with $v_0 = e$ and $v_j = v$. Then for any $0 \leq i \leq j$, we have $v_{i-1} v_i^{-1} \in \{a_i^2 \mid 1 \leq i \leq k\}$. Noting that the elements in $\{a_i^2 \mid 1 \leq i \leq k\}$ are all involutions and commute with each other, it follows that $v = a_{i_1}^2 a_{i_2}^2 \dots a_{i_j}^2$ for some $1 \leq i_1, \dots, i_j \leq k$. Consequently, $v \in \langle a_i^2 \mid 1 \leq i \leq k \rangle$. Set $H = \langle a_i^2 \mid 1 \leq i \leq k \rangle$. The above argument gives that $V(A) \subseteq H$. Clearly, $H \cong \mathbb{Z}_2^k$, and the subgraph of $\Gamma_{k,\ell}$ induced by H is just the Cayley graph $\text{Cay}(H, \{a_i^2 \mid 1 \leq i \leq k\})$, which is isomorphic to Q_k .

For any two distinct vertices $u, v \in V(A)$, if u, v have a common neighbor, say w , in $V(\Gamma_{k,\ell}) \setminus H$, then $w = s_1 u = s_2 v$ for some $s_1, s_2 \in \Omega_{k,\ell} \setminus \{a_i^2 \mid 1 \leq i \leq k\}$, and hence $s_1 s_2 = uv^{-1} \in \langle a_i^2 \mid 1 \leq i \leq k \rangle$. However, it is easy to check that the product of any two distinct elements in $\Omega_{k,\ell} \setminus \{a_i^2 \mid 1 \leq i \leq k\}$ can not be in $\langle a_i^2 \mid 1 \leq i \leq k \rangle$. Thus, $s_1 = s_2$. Since $w = s_1 u = s_2 v$, one has $u = v$, a contradiction. Thus, any two distinct vertices of A do not share a common neighbor in $V(\Gamma_n) \setminus H$. It follows that $|N_{V(\Gamma_{k,\ell}) \setminus H}(A)| = |A|(n - k)$, and so

$$|N_H(A)| = f_n(g) - |A|(n - k) = f_k(g).$$

By Lemma 2.6, $Q_k[A] = \Gamma_{k,\ell}[A]$ is a star, as required.

Case 2 There is an edge $\{x, y\}$ of A such that $xy^{-1} \notin \{a_i^2 \mid 1 \leq i \leq k\}$.

Let $xy^{-1} = b$. Then $b \in \Omega_{k,\ell}$ and $x = by$. Set $H = \langle \Omega_{k,\ell} - b \rangle$, $\Gamma_{k,\ell}^0 = \text{Cay}(H, \Omega_{k,\ell} - b)$ and $\Gamma_{k,\ell}^1 = \Gamma_{k,\ell}[Hb]$. It is easy to see that $H \cong G_{k-1,\ell+2}$ or $G_{k,\ell-1}$, and $\Gamma_{k,\ell}^0 \in \mathcal{G}_{n-1}$. In view of the fact that the map $R(b) : v \mapsto vb, \forall v \in G_{k,\ell}$ is an automorphism of $\Gamma_{k,\ell}$, it follows that $\Gamma_{k,\ell}^1 = \Gamma_{k,\ell}[Hb] \cong \Gamma_{k,\ell}^0$. Also, since $b \notin \{a_i^2 \mid 1 \leq i \leq k\}$, H has index 2 in $G_{k,\ell}$, and so H has two cosets, namely, H and Hb , in $G_{k,\ell}$. Without loss of generality, assume that $y \in H = V(\Gamma_{k,\ell}^0)$. Then $x = by \in bH = Hb = V(\Gamma_{k,\ell}^1)$. So $U_i = V(\Gamma_{k,\ell}^i) \cap V(A)$ is non-empty for $i = 0, 1$. Without loss of generality, assume that $|U_0| \leq |U_1|$. Set $|U_0| = N$. Then $N \leq \lfloor \frac{g+1}{2} \rfloor$. By Proposition 2.3, $|N_{\Gamma_{k,\ell}^0}(U_0)| \geq f_{n-1}(N-1)$ and $|N_{\Gamma_{k,\ell}^1}(U_1)| \geq f_{n-1}(g-N)$. Note that $N_{\Gamma_{k,\ell}^0}(U_0) \cup N_{\Gamma_{k,\ell}^1}(U_1) \subseteq N_{\Gamma_{k,\ell}}(V(A))$ and $N_{\Gamma_{k,\ell}^0}(U_0) \cap N_{\Gamma_{k,\ell}^1}(U_1) = \emptyset$. It follows that $|N_{\Gamma_{k,\ell}^0}(U_0)| + |N_{\Gamma_{k,\ell}^1}(U_1)| \leq |N_{\Gamma_{k,\ell}}(V(A))| = f_n(g)$, and hence

$$\begin{aligned} 0 &\geq |N_{\Gamma_{k,\ell}^0}(U_0)| + |N_{\Gamma_{k,\ell}^1}(U_1)| - |N_{\Gamma_{k,\ell}}(V(A))| \\ &\geq f_{n-1}(N-1) + f_{n-1}(g-N) - f_n(g) \\ &= -(N-1)(N-g). \end{aligned} \tag{2}$$

Remember that $1 \leq N \leq \lfloor \frac{g+1}{2} \rfloor$. If $N > 1$, then $N < g$ and so $0 \geq -(N-1)(N-g) > 0$, a contradiction. Thus, $N = 1$ and so $|U_1| = g$. Let $U_0 = \{v\}$. Since v has only one neighbor in V_1 , v is a pendant vertex of A , and so $\Gamma_{k,\ell}[U_1]$ is connected. Again, since $N = 1$, by the above equation (2), we have $|N_{\Gamma_{k,\ell}^1}(U_1)| = f_{n-1}(g-1)$ and $|N_{\Gamma_{k,\ell}}(V(A))| = |N_{\Gamma_{k,\ell}^0}(U_0)| + |N_{\Gamma_{k,\ell}^1}(U_1)|$. Then $|N_{\Gamma_{k,\ell}}(U_1)| = |N_{\Gamma_{k,\ell}^1}(U_1)| + |N_{\Gamma_{k,\ell}^0}(U_1)| = f_n(g-1)$. By the induction hypothesis, we have $\Gamma_{k,\ell}^1[U_1]$ is a star or a 3-path. The equation $|N_{\Gamma_{k,\ell}}(V(A))| = |N_{\Gamma_{k,\ell}^0}(U_0)| + |N_{\Gamma_{k,\ell}^1}(U_1)|$ implies that $N_{\Gamma_{k,\ell}^0}(U_1 - \alpha) \subseteq N_{\Gamma_{k,\ell}^0}(v)$, where α is the neighbor of v in $\Gamma_{k,\ell}^1$.

Suppose that $\Gamma_{k,\ell}[U_1]$ is a 3-path $u_1u_2u_3u_4$. Assume that the neighbor of u_i in $\Gamma_{k,\ell}^0$ is w_i for each $1 \leq i \leq 4$. Then $v \in \{w_i \mid 1 \leq i \leq 4\}$.

Let $v = w_1$ or w_4 . Without loss of generality, assume that $v = w_1$ (see Figure 3). Suppose that u_1 and w_3 has a common neighbor, say x , such that $x \neq v$. Since the edges

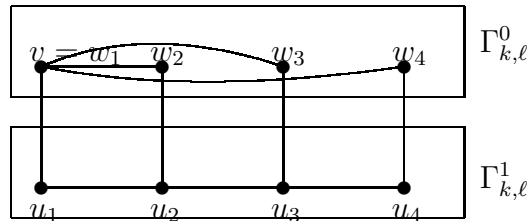


Figure 3: Illustration of the proof for the case $v = w_1$

between $\Gamma_{k,\ell}^0$ and $\Gamma_{k,\ell}^1$ are vertex-disjoint, we must have $x = u_3$, forcing (u_1, u_2, u_3) is a triangle, contrary to the fact that $\Gamma_{k,\ell}$ has girth 4 (Proposition 2.4). Thus, v is the unique common neighbor of u_1 and w_3 . By Lemma 5.2, $\{u_1, w_3\} = \{b_iv, a_ib_iv\}$ for some $1 \leq i \leq s$.

Clearly, v is also a common neighbor of u_1 and w_4 . By Lemma 5.2, if u_1 and w_4 have only one common neighbor, then we also have $\{u_1, w_4\} = \{b_i v, a_i b_i v\}$, which is impossible. Thus, u_1 and w_4 must have two common neighbors, and so u_4 must be a common neighbor of u_1 and w_4 . Thus, (u_1, u_2, u_3, u_4) is a 4-cycle. By Proposition 2.4, any two vertices of $\Gamma_{k,\ell}$ have at most two common neighbors. It follows that $|N_{\Gamma_{k,\ell}^1}(U_1)| = 4(n-3) > f_{n-1}(3) = 4n-13$, a contradiction.

Let $v = w_2$ or w_3 . Without loss of generality, assume that $v = w_2$ (see Figure 4). Suppose that w_4 and u_2 have a common neighbor, say w , different from v . If $w \in V(\Gamma_{k,\ell}^0)$,

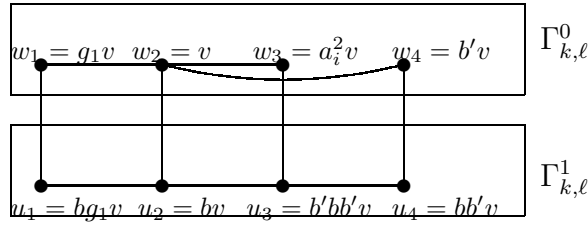


Figure 4: Illustration of the proof for the case $v = w_2$

then since u_2 has a unique neighbor in $V(\Gamma_{k,\ell}^0)$, we must have $w = v$, a contradiction. If $w \in V(\Gamma_{k,\ell}^1)$, then since w_4 has a unique neighbor in $V(\Gamma_{k,\ell}^1)$, we must have $w = u_4$. It follows that $(u_2, u_3, u_4 = w)$ is a triangle, contrary to the fact that $\Gamma_{k,\ell}$ has girth 4 (Proposition 2.4). Thus, v is the unique common neighbor of w_4 and u_2 . With a similar argument, we have u_4 is the unique common neighbor of u_3 and w_4 , and w_4 is the unique common neighbor of w_2 and u_4 .

By Lemma 5.2, $\{w_4, u_2\} = \{b_i v, a_i b_i v\}$ for some $1 \leq i \leq k$. Let $w_4 = b' v$ for some $b' \in \Omega_{k,\ell}$. It is easy to see that for each $u \in H = V(\Gamma_{k,\ell}^0)$, bu is the unique neighbor of u in $Hb = V(\Gamma_{k,\ell}^1)$. Thus, $u_2 = b w_2 = b v$, and so $\{b' v, b v\} = \{w_4, u_2\} = \{b_i v, a_i b_i v\}$. It follows that $\{b, b'\} = \{b_i, a_i b_i\}$. Also, $u_4 = b w_4 = b b' v$. As u_4 is the unique common neighbor of u_3 and w_4 , by Lemma 5.2, $\{w_4, u_3\} = \{b u_4, b' u_4\} = \{b' v, b' b b' v\}$ because $w_4 = b u_4$. Consequently, $u_3 = b' u_4 = b' b b' v$, and so $w_3 = b u_3 = (b b')^2 v = a_i^2 v$ (note that $b b' = a_i$ or a_i^{-1}). Set $w_1 = g_1 v$ with $g_1 \in \Omega_{k,\ell}$. Then $u_1 = b g_1 v$. Note that $\{w_3, w_4, u_2\} = \{a_i^2 v, a_i b_i v, b_i v\}$. So, $g_1 \notin \{a_i^2, a_i b_i, b_i\}$. Since $g_1 \in \Omega_{k,\ell}$, one has $g_1 \notin \langle a_i, b_i \rangle$, and so g_1 commutes with the elements in $\langle a_i, b_i \rangle$. In particular, g_1 commutes with b and b' . Since $u_2 = b v$ is adjacent to $u_3 = b' b b' v$, there exists an $s \in \Omega_{k,\ell}$ such that $s u_2 = u_3$, namely, $s b = b' b b'$. It follows that $s = (b' b)^2 = a_i^2$. Now one may see that $u_1 (= g_1 u_2)$ and $u_3 (= s u_2)$ have a common neighbor $g_1 s u_2$ which is different from u_2 . Moreover, $u_2 (= a_i^2 u_3)$ and $u_4 (= b' u_3)$ also have a common neighbor $a_i^2 b' u_3$ which is different from u_3 . By Proposition 2.4, any two vertices of $\Gamma_{k,\ell}$ have at most two common neighbors. Since $|N_{\Gamma_{k,\ell}^1}(U_1)| = f_{n-1}(3) = 4n-13$, u_1 and u_4 must have a unique common neighbor, say w . Again by Lemma 5.2, we have $\{u_1, u_4\} = \{b_j w, a_j b_j w\}$, and so $u_1 u_4^{-1} \in \{a_j, a_j^{-1}\}$ for some $1 \leq j \leq k$. This implies that $u_1 u_4^{-1}$ has order 4. However, $u_1 u_4^{-1} = b g_1 v (b b' v)^{-1} = g_1 a_i^2 b'$ is an involution, a contradiction.

Now we assume that $\Gamma_{k,\ell}[U_1]$ is a star with vertex-set $U_1 = \{u, u_1, \dots, u_{g-1}\}$ and edge-

set $E(\Gamma_{k,\ell}[U_1]) = \{\{u, u_i\} \mid 1 \leq i \leq g-1\}$. If v is adjacent to u , then A is a star. Assume that v is adjacent to some u_i . Without loss of generality, let $i = 1$. Assume that the neighbor of u_i in $\Gamma_{k,\ell}^0$ is w_i for each $1 \leq i \leq g-1$, and the neighbor of u in $\Gamma_{k,\ell}^0$ is w . Then $v = w_1$. If $g \leq 2$, then A is also a star. If $g = 3$, then A is a 3-path. Suppose $g > 3$ (see Figure 5). Since $N_{\Gamma_{k,\ell}^0}(U_1 - u_1) \subseteq N_{\Gamma_{k,\ell}^0}(v)$, for any $u_j \in U_1 - \{u, u_1\}$, the neighbor w_j of u_j in $\Gamma_{k,\ell}^0$ is also adjacent to v , and hence (v, u_1, u, u_j, w_j) is a cycle of length 5. Suppose that

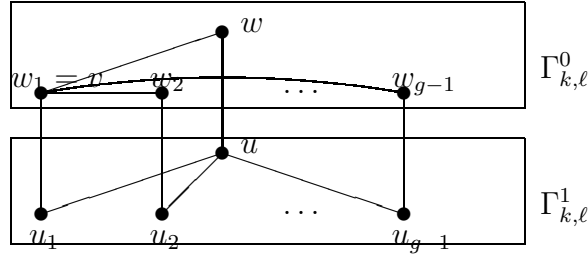


Figure 5: Illustration of the proof for the case $v = w_1$

v is not the unique common neighbor of u_1 and w_j for some $2 \leq j \leq g-1$. Then u_1 and w_j have another common neighbor say x , different from v . Note that the edges between $V(\Gamma_n^0)$ and $V(\Gamma_n^1)$ form a perfect matching. If $x \in V(\Gamma_{k,\ell}^0)$, then $x = v$, a contradiction. If $x \in V(\Gamma_{k,\ell}^1)$, then $x = u_j$, and so (u_1, u, u_j) is a triangle, contrary to the fact that $\Gamma_{k,\ell}$ has girth 4 (Proposition 2.4). Thus, v is the unique common neighbor of u_1 and w_j for each $2 \leq j \leq g-1$. By Lemma 5.2, we have $\{u_1, w_j\} = \{b_{i_j}v, a_{i_j}b_{i_j}v\}$ for some $0 \leq i_j \leq k$. Since $g > 3$, one has $g-1 > 2$, and hence $\{u_1, w_2\} = \{b_{i_2}v, a_{i_2}b_{i_2}v\}$ and $\{u_1, w_{g-1}\} = \{b_{i_{g-1}}v, a_{i_{g-1}}b_{i_{g-1}}v\}$. Since $u_1 \in \{b_{i_{g-1}}v, a_{i_{g-1}}b_{i_{g-1}}v\} \cap \{b_{i_2}v, a_{i_2}b_{i_2}v\}$, by the structure of the group G_n we must have $i_2 = i_{g-1}$. It follows that $\{u_1, w_2\} = \{u_1, w_{g-1}\}$, and hence $w_2 = w_{g-1}$, a contradiction. \square

Theorem 5.4 *If $2k+1 \leq g \leq 3k-4$, then $\kappa_g(\Gamma_{k,0}) > f_{3k}(g)$.*

Proof By Theorem 3.3, $\kappa_g(\Gamma_{k,0}) \geq f_{3k}(g)$. Suppose on the contrary that $\kappa_g(\Gamma_{k,0}) = f_{3k}(g)$. Let S be a minimum R_g -cutset of $\Gamma_{k,0}$. By Theorem 3.3, $\Gamma_{k,0} - S$ has two components, one of which, say A , has $g+1$ vertices. By the minimality of S , we see that $S = N_{\Gamma_{k,0}}(V(A))$. Since $|S| = \kappa_g(\Gamma_{k,0}) = f_{3k}(g)$, by Lemma 5.3, A is isomorphic to $K_{1,g}$ or P_3 . Since $2k+1 \leq g \leq 3k-4$, one has $k \geq 5$, and so $g+1 \geq 12$. It follows that $A \cong K_{1,g}$. By the vertex-transitivity of $\Gamma_{k,0}$, we may assume that the identity element e of $G_{k,0}$ is a vertex of A , and $V(A) = \{e, s_1, s_2, \dots, s_g\}$. By Proposition 2.4, any two vertices of $\Gamma_{k,0}$ have at most two common neighbors. Since $|N_{\Gamma_{k,0}}(V(A))| = f_{3k}(g)$, by an easy calculation, we see that for any $s_i, s_j \in V(A) - \{e\}$, they have exactly two common neighbors. Clearly, $V(A) - \{e\} \subseteq \Omega_{k,0}$. Since $g \geq 2k+1$, there must exist $1 \leq i \leq k$ such that $a_i b_i, b_i \in V(A)$. So, $a_i b_i$ and b_i have two common neighbors. Clearly, e is a common neighbor of them. Let $u \neq e$ be another common neighbor of $a_i b_i$ and b_i . Then

$u = xa_i b_i = yb_i$ for some $x, y \in \Omega_{k,0}$. It follows that $xy = a_i$. In view of the fact that $\Omega_{k,0} = \{a_i^2, b_i, a_i b_i \mid 1 \leq i \leq k\}$, we must have $x = a_i b_i$ and $y = b_i$, forcing $u = e$, a contradiction. \square

Remark 2 This theorem implies that [30, Theorem 3.4], which states that the g -extra connectivity of an n -dimensional HL-network G is $f_n(g)$ for $0 \leq g \leq n-4$, is not true. In fact, in the proof of [30, Theorem 3.4], the authors first assumed that $G = G_0 \oplus G_1$, and then picked a subgraph, say A , of G such that $A \cong K_{1,g}$ and $A \subseteq G_0$. Then they claimed that “it is straightforward to see that $|N_G(A)| = n(g+1) - \frac{g(g+3)}{2}$.” However, from the proof of Theorem 5.4 we see that this claim is not true.

6 Conclusion

As one of the novel factors for measuring the reliability and fault tolerance of networks, the g -extra connectivity of HL-networks have also been studied by some authors. From [28] we see that $\kappa_g(Q_n) = f_n(g) = n(g+1) - \frac{1}{2}g(g+3)$ where $n \geq 4$ and $0 \leq g \leq n-3$. In view of the fact that the hypercubes Q_n belong to the class of HL-networks, an interesting problem is: For an n -dimensional HL-network G , does $\kappa_g(G) = f_n(g)$ hold for $n \geq 4$ and $0 \leq g \leq n-3$?

For this problem, some partial answers have been known. Several authors proved that the answer is positive for the case when $g \leq 3$ (see [14, 29, 25, 4]). In this paper, a subclass of n -dimensional HL-networks with g -extra connectivity greater than $f_n(g)$ is presented, and so a negative answer to the above problem is given.

Moreover, we also prove that for $n \geq 5$ and $0 \leq g \leq n-3$, if n -dimensional HL-network X_n has a R_g -cutset S with $|S| \leq f_n(g)$, then $|S| = f_n(g)$ and X_n is hyper- κ_g . This enables us to obtain a lower bound on $\kappa_g(G)$, namely, $\kappa_g(X_n) \geq f_n(g)$. In addition, we also give a sufficient condition for the lower bound being attainable. Applying this, we first present a short proof for the main results of [4] which shows that the answer to the above problem is positive for the case when $g = 2$ or 3 , and then determine the g -extra connectivity of the varietal hypercubes for some specific g .

In view of the above facts, an interesting problem is: Determine the smallest g such that $\kappa_g(G) = f_n(g)$ holds for all n -dimensional HL-networks G . This is a topic for our future effort.

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